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Finding All Solutions Related to Steinmetz's Theorem

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I. INTRODUCTION

For all F meromorphic in the complex plane let $T(r, F)$ denote the Nevanlinna characteristic function of F. Suppose below that the h_j and the F_j are each meromorphic in the complex plane, for $j = 1, 2, ..., n$; that g is entire; that not all of the F_j are zero; and that

$$\sum_j T(r, h_j) = O(T(r, g)).$$

Under these conditions Steinmetz [1] proved the following theorem:

THEOREM A. *If*

$$\sum_{j=1}^{n} F_j(g)\, h_j = 0,$$

then there exist polynomials $p_j(z)$, not all zero, such that

$$\sum_{j=1}^{n} p_j(g) h_j = 0. \tag{1}$$

Steinmetz's proof made use of Nevanlinna's First and Second Fundamental Theorems. It was shown by the present authors in [2] that Steinmetz's proof could be simplified and that the use of the Second Fundamental Theorem could be avoided.

II. MAIN RESULTS

Expanding upon the arguments in the simpler proof, the authors were able to prove in [3] the following generalization of Steinmetz's Theorem:

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Let $n \geqslant 2$ be an integer. Suppose that g is a nonconstant entire function, and that the Ψ_j, for $1 \leqslant j \leqslant n$, are entire functions. Suppose that f_j and h_j, for $1 \leqslant j \leqslant n$, are meromorphic functions, and that there exist positive reals A and B such that

$$A \leqslant 1, \qquad B < \frac{1}{64n^4 A},$$

$$\sum_{j=1}^{n} T(r, h_j) \leqslant AT(r, g),$$

and

$$\sum_{j=1}^{n} T(r, \Psi_j) \leqslant BT(r, g).$$

Suppose that each f_j is analytic at $z = 0$ and that $f_1(0) \neq 0$.

THEOREM B. *Under the above conditions if*

$$\sum_{j=1}^{n} f_j(g\Psi_j)h_j(z) = 0,$$

then there exist n functions $p_j(z, w)$, not all zero, where each $p_j(z, w)$ is a polynomial in the variables $\Psi_1, \Psi_2, ..., \Psi_n$ and w such that the inner product

$$\sum_{j=1}^{n} p_j(z, g)h_j(z) = 0.$$

In this paper it will be shown how further extensions of the elementary arguments used in [3] can be used to determine all sets p_j that appear in the conclusion of Theorem A; that is, all sets satisfying Eq. (1). We prove the following result.

THEOREM. *Suppose that we are given $h_1(z), h_2(z), ..., h_n(z)$ as described above. Let P be the vector space of all n-tuples of meromorphic functions $(F_1, ..., F_N)$ such that*

$$(F_1(g), F_2(g), ..., F_n(g)) \cdot (h_1(z), h_2(z), ..., h_n(z)) = 0.$$

One can explicitly construct a basis of P. Generally, one can write

$$(F_1, F_2, ..., F_n) = \sum_k G_k(P_{1k}, P_{2k}, ..., P_{nk}),$$

for a set of meromorphic functions G_k, *where the* P_{ik}, $i = 1, 2, ..., n$; $k = 1, 2, ..., m$ *are polynomials in* z *each of degree at most equal to*

$$d = 2(n-1)\left[\liminf_{r\to\infty}\left(\sum_j T(r, h_j)\right)(T(r, g))^{-1}\right] + 1,$$

where [ ] *denotes the greatest integer function.*

COROLLARY. *Given an entire function* g, *let* F_g, *denote the field of all functions of the form* $f(g)$, *where* f *is meromorphic in the plane. Suppose that* $h_1, ..., h_n$, *are* $n > 1$ *nonzero functions each meromorphic in the plane. Where* d *is as defined above, the* h_i *are linearly dependent over* F_g *iff the following Wronskian*

$$W(h_1, gh_1, g^2h_1, ..., g^dh_1; ...; h_n, gh_n, g^2h_n, ..., g^dh_n)$$

vanishes identically.

Proof of the Corollary. If there is a dependency over F_g then the hypotheses of this theorem are satisfied, so by its conclusion this Wronskian must vanish. The converse is trivial.

Proof of the Theorem. Suppose that

$$\sum_{j=1}^{n} p_{jk}(g)h_j = 0$$

is a maximal set of $m < n$ independent polynomial dependence relations connecting the h_j. We suppose, as we shall show, without loss of generality, that these polynomial dependencies are in upper triangular form. That is only zeros are below the main diagonal of the matrix of coefficients. There is no loss of generality in using rational function coefficients. Pick one of these not identically zero equations arbitrarily. Let the first variable be chosen as one for which the first equation has a nonzero coefficient p_j, now call it p_1 and the corresponding h_j is now called h_1. Eliminate h_1 in each of the other forms. Continuing, we arrive at an upper triangular form. Now append the original dependence relation $\sum_j F_j(g)h_j$ to the set of dependencies over the polynomials. First, suppose that $\sum_j F_j(g)h_j$ is not linearly dependent upon these polynomial dependencies. Then we could extend our upper triangular matrix by applying the triangulization procedure above to the additional form, obtaining a not identically zero (probably generally nonpolynomial) dependence relation. By Steinmetz's Theorem this last dependency may be replaced by a polynomial dependency in a nonempty subset of the same h_j's. Possibly resubscripting the F_j's in this latter form, we have now produced $m+1$ polynomial dependencies in upper triangular form; hence, they are independent, contradicting the maximality of m. Thus, the form $\sum_{j=1}^{n} F_j(g)h_j$ is linearly dependent on the polynomial dependencies over the field of rational function of g.

It follows that forming the matrix of the coefficients of a maximal set of polynomial dependencies, each of the form $\sum_{j=1}^{n} p_{jk}(g)h_j$ plus the original dependence relation $\sum_{j=1}^{n} F_j(g)h_j$, the rows are dependent. This condition may be written as

$$(F_1(z), ..., F_n(z)) = \sum_k G_k(z)(p_{1k}(z), ..., p_{nk}(z)),$$

for a set of meromorphic functions $G_k(z)$.

Suppose that we are given a maximal set of polynomial dependencies as above in triangular form. The proof of Steinmetz's theorem in [2] can be made to yield the bound (on the degrees of the polynomial coefficients in the dependency) of the least integer greater than

$$d = 2(n-1)\left[\liminf_{r\to\infty}\left(\sum_j T(r, h_j)\right)(T(r, g))^{-1}\right].$$

We shall refer to d as the Steinmetz bound. (Here is how to obtain the bound. In [2], note from Lemma 1 that mM $(=n-1)$ is a bound on the degrees of the coefficient polynomials. To bound M, use [2, (5)] to see that $m(r, 1/H_M(z)) \leqslant \sum_{i=1}^{n} T(r, h_i(z)) + K$, place this bound in the final inequality in [2], and take $\liminf_{r\to\infty}$ of the resulting expression.) We wish to see that there exists a set of polynomial dependencies in upper triangular form of the same dimension and with coefficients bounded in degree by the Steinmetz bound. Suppose not and that the respective dimensions are m_1 and m_2, where $m_1 < m_2$. We may form a linear combination of m_1 of the set of m_2 independent polynomial forms that when added to the m_1 forms above extends the upper triangularization, possibly after resubscripting. Of course, the added form may well be of much higher degree. However, applying Steinmetz's Theorem to this last form (polynomials are meromorphic functions after all!) produces a possibly new, nonzero polynomial dependency with coefficients satisfying the Steinmetz bound. Possibly after resubscripting, this new dependency will also extend the upper triangularization form, with coefficients satisfying the Steinmetz bound. This contradiction shows that a spanning set of the polynomial dependencies exist with coefficients satisfying the Steinmetz bound. Suppose that we have available the expansions of the $F_j(x)$'s at $x=0$, following the simpler proof of Steinmetz's Theorem in [2]. We may start with undetermined coefficients and attempt to determine polynomial coefficients such that the form $\sum_{j=1}^{n} F_j p_j$ vanishes to a high power at $x=0$. We know from the proof of Steinmetz's theorem in [2] that if the order of vanishing of this form is high enough, then dependence actually holds. Thus one can make this an effective calculation and obtain all polynomial dependencies satisfying the Steinmetz bound. Thus, a spanning set of polynomial dependencies can be effectively calculated. This completes the proof of our Theorem.

III. Concluding Remarks

The corollary of Steinmetz's Theorem in [1] says that the F_j's are linearly dependent over the polynomials. This follows trivially from our theorem, namely from the result that

$$(F_1(z), ..., F_n(z)) = \sum_k G_k(z)(p_{1k}(z), ..., p_{nk}(z)),$$

since we may write down dependence relations. We may take the p_{ik} to satisfy the Steinmetz bound. Thus the dependence relations here are bounded in degree; however, other dependencies may possibly exist among the F_j's since the G_k could be dependent over the polynomials and, furthermore, the degree of the dependence relation could be high.

References

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